

The conformal dilatation and Beltrami forms over quadratic field extensions

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Contents

Introduction

1. Bilinear and quadratic forms
2. The norm and the trace
3. Quadratic forms in dimension 1 over \mathbf{K}
4. The structure of a \mathbf{K} -vector space on $\mathbf{A}(V)$
5. Conformal structures
6. Anti-linear maps
7. The structure of $\mathrm{Hom}_{\mathbf{a}}(V, V)$
8. The conformal dilatation and the Beltrami forms
9. Identification of $\mathbb{M}(V)$ with $\mathrm{Hom}_{\mathbf{a}}(V, V)$
10. Comparing $c(f)$ and μ_f

References

Introduction

The motivation. The present paper grew out of the desire to present an invariant treatment of the geometric interpretation of the complex dilatation suggested in [I]. The main part of [I] is devoted to the real linear maps $\mathbf{C} \rightarrow \mathbf{C}$ and heavily uses the standard basis $1, i$ of \mathbf{C} as a real vector space. Only the last section of [I] is devoted to the general case of real linear maps $V \rightarrow W$, where V, W are two complex vector spaces of dimension 1, and this case is treated by a reduction to the case $\mathbf{C} \rightarrow \mathbf{C}$. This approach served well to the goals of the paper [I]; see [I], the end of the Introduction.

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A recent paper of M. Atiyah [A] reminded the author that a truly invariant approach requires more than just dealing with maps $V \rightarrow W$ from the very beginning. Namely, to quote M. Atiyah, “one should not distinguish between i and $-i$ whereas one can distinguish between 1 and -1 ”. An attempt to follow this maxim quickly lead to the realization that one should distinguish between -1 as a purely additive notion and i^2 .

Since $i^2 = -1$ anyhow, the only sensible way to distinguish -1 and i^2 is to abandon the axiom $i^2 + 1 = 0$ and replace \mathbf{C}, \mathbf{R} by two fields \mathbf{K}, \mathbf{k} such that \mathbf{K} is an extension of \mathbf{k} of degree $[\mathbf{K} : \mathbf{k}] = 2$. Since quadratic forms behave differently in characteristic 2, it is reasonable to assume that the characteristic of \mathbf{k} is $\neq 2$.

This paper is devoted to an analogue of the theory of [I] in this situation, independent on any choices of bases. Working with a general field extension \mathbf{K}/\mathbf{k} does not lead to any new difficulties compared to the classical case $\mathbf{K}/\mathbf{k} = \mathbf{C}/\mathbf{R}$, but only clarifies the theory. In the rest of the introduction is devoted to an outline of this analogue.

Conformal structures and conformal dilatation. For a vector space V over \mathbf{k} we denote by $Q(V)$ the vector space of all quadratic forms on V . A *conformal structure* on V is a non-zero quadratic form on V considered up to multiplication by non-zero elements of \mathbf{k} , i.e. a point in the projective space $\mathbb{P}Q(V)$ associated with $Q(V)$. If $\dim V = 2$, then $\mathbb{P}Q(V)$ is a projective plane and the set $\mathcal{C}(V) \subset \mathbb{P}Q(V)$ of conformal structures corresponding to degenerate quadratic forms is a *conic* in $\mathbb{P}Q(V)$. In other terms, $\mathcal{C}(V)$ is defined by a homogeneous equation of degree 2.

This paper is concerned mostly with vector spaces V of dimension 1 over \mathbf{K} , considered as a vector spaces over \mathbf{k} . There is a *canonical conformal structure* on any such V , denoted by c_V . It is invariant under the multiplication by non-zero elements of \mathbf{K} .

Let $\mathcal{A}(V)$ be the projective line polar* to c_V with respect to the conic $\mathcal{C}(V)$. Then

$$\mathcal{M}(V) = \mathbb{P}Q(V) \setminus \mathcal{A}(V)$$

is an affine plane containing c_V . One can turn $\mathcal{M}(V)$ into a vector space over \mathbf{k} by designating c_V as its zero vector. The structure of a vector space of dimension 1 over \mathbf{K} on V allows to extend this structure of a vector space over \mathbf{k} on $\mathcal{M}(V)$ to a structure of a vector space over \mathbf{K} on $\mathcal{M}(V)$ in a canonical manner. There is a canonical quadratic form $\mathcal{D} : \mathcal{M}(V) \rightarrow \mathbf{k}$ equal to 1 on the conic $\mathcal{C}(V)$.

Let $f : V \rightarrow W$ be a map linear over \mathbf{k} between vector spaces V, W of dimension 1 over \mathbf{K} . A natural measure of the distortion of the canonical conformal structure by f is the pull-back $c(f) = f^*c_W \in \mathbb{P}Q(V)$, called the *conformal dilatation* of f .

* In the main part of the paper the notion of polarity is not used and $\mathcal{A}(V)$ is defined differently. Theorem 3.3 implies that the two definitions are equivalent.

Anti-linear maps and Beltrami forms. Since $[\mathbf{K} : \mathbf{k}] = 2$, there is only one non-trivial automorphism of \mathbf{K} fixed on \mathbf{k} , which is denoted by $z \mapsto \bar{z}$.

Let V, W be vector spaces of dimension 1 over \mathbf{K} . A map $g : V \rightarrow W$ is said to be *anti-linear* if it is linear over \mathbf{k} and $g(zv) = \bar{z}g(v)$ for all $z \in \mathbf{K}, v \in V$. Such maps form a vector space $\text{Hom}_a(V, W)$ over \mathbf{K} . There is a canonical quadratic form

$$\mathcal{D} : \text{Hom}_a(V, V) \rightarrow \mathbf{k}.$$

In view of Lemma 7.1 it can be defined by $\mathcal{D}(g) = -\det g$, where the determinant is taken over \mathbf{k} .

If a map $f : V \rightarrow W$ is linear over \mathbf{k} , then $f = Lf + Af$, where Lf is linear over \mathbf{K} and Af is anti-linear. Moreover, the maps Lf, Af are uniquely determined by f . If $Lf \neq 0$, then Lf is invertible and the composition

$$\mu_f = (Lf)^{-1} \circ (Af) \in \text{Hom}_a(V, V).$$

is a natural measure of failure of f to be linear over \mathbf{K} , called the *Beltrami form* of f .

Comparing the conformal dilatation and Beltrami forms. The results of [I] suggest that $c(f)$ and μ_f should be related by an analogue of the map relating the Klein and the Poincaré models of the hyperbolic plane. As it turns out, this is indeed the case.

There is a canonical isomorphism of $\mathbb{M}(V)$ and $\text{Hom}_a(V, V)$ as vector spaces over \mathbf{K} . Moreover, this isomorphism turns the quadratic forms \mathbb{D}, \mathcal{D} one into the other. One can use this isomorphism to identify $\mathbb{M}(V)$ and $\text{Hom}_a(V, V)$.

Let $f : V \rightarrow W$ be a map linear over \mathbf{k} . The map f is called *regular* if $c(f) \in \mathbb{M}(V)$, and *exceptional* otherwise. The exceptional maps are a new feature of the general case. If the extension \mathbf{K}/\mathbf{k} resembles enough the classical case of \mathbf{C}/\mathbf{R} , then all non-zero maps f are regular. This is the case if the field \mathbf{k} is ordered and \mathbf{K} is obtained by adjoining a root of a *negative* element to \mathbf{k} , as it easily follows from Theorem 10.2.

Suppose that $Lf \neq 0$. If $1 + \mathcal{D}(\mu_f) \neq 0$, then f is regular and

$$c(f) = \frac{2\mu_f}{1 + \mathcal{D}(\mu_f)}$$

after the identification of $\mathbb{M}(V)$ with $\text{Hom}_a(V, V)$. See Theorem 10.2. A simple calculation (see [I], Corollary 3.3) shows that for $\mathbf{K} = \mathbf{C}$, $\mathbf{k} = \mathbf{R}$, and $V = \mathbf{C}$ this relation between $c(f)$ and μ_f is the same as in [I].

If $1 + \mathcal{D}(\mu_f) = 0$, then f is exceptional. In this case the relation between $c(f)$ and μ_f is even simpler. See Theorem 10.3.

1. Bilinear and quadratic forms

Bilinear forms, their matrices and determinants. Let us fix a vector space V over k of finite dimension n and a basis v_1, v_2, \dots, v_n of V . A bilinear form $\varphi: V \times V \rightarrow k$ gives rise to an $n \times n$ matrix $M(\varphi) = (M_{ij}(\varphi))$ with the entries

$$M_{ij}(\varphi) = \varphi(v_i, v_j),$$

called *the matrix of φ* with respect to the basis v_1, v_2, \dots, v_n . The determinant

$$\det \varphi = \det M(\varphi)$$

is called the *determinant* of φ (with respect to the basis v_1, v_2, \dots, v_n).

Change of basis. If v'_1, v'_2, \dots, v'_n is some other basis of V , then

$$v'_i = a_{i1}v_1 + a_{i2}v_2 + \dots + a_{in}v_n$$

for some invertible matrix $A = (a_{ij})$ and

$$M'(\varphi) = AM(\varphi)A^t,$$

where $M'(\varphi)$ is the matrix of φ with respect to the basis v'_1, v'_2, \dots, v'_n , and A^t is the transposed matrix of A . Therefore

$$\det' \varphi = \det \varphi \cdot (\det A)^2,$$

where $\det' \varphi$ is the determinant of φ with respect to the basis v'_1, v'_2, \dots, v'_n .

Bilinear forms and linear maps. If $f: V \rightarrow V$ is a linear map, then

$$f(v_i) = f_{i1}v_1 + f_{i2}v_2 + \dots + f_{in}v_n,$$

where $F = (f_{ij})$ is the matrix of f in the basis v_1, v_2, \dots, v_n . Together with the form φ the map f gives rise to a new bilinear form, the *left product*

$$f \cdot \varphi : (v, w) \mapsto \varphi(f(v), w),$$

of f and φ . The matrix $M(f \cdot \varphi)$ of $f \cdot \varphi$ is equal to $FM(\varphi)$, and hence

$$(1.1) \quad \det(f \cdot \varphi) = \det f \cdot \det \varphi,$$

where $\det f = \det F$ is independent on the choice of the basis.

Non-degenerate bilinear forms. A bilinear form φ gives rise to a linear map

$$\widehat{\varphi} : V \longrightarrow \text{hom}(V, \mathbf{k})$$

taking $v \in V$ to the map $w \mapsto \varphi(v, w)$. The form φ is said to be *non-degenerate* if $\widehat{\varphi}$ is an isomorphism. As is well known, this condition is equivalent to $\det \varphi \neq 0$.

Suppose that φ, β are bilinear forms on V and φ is non-degenerate. By the previous paragraph, for every $v \in V$ the homomorphism $w \mapsto \beta(v, w)$ is equal to

$$w \mapsto \varphi(f(v), w)$$

for a unique $f(v) \in V$. A trivial verification shows that f is linear and hence $\beta = f \cdot \varphi$.

Quadratic forms. A bilinear form $\varphi : V \times V \longrightarrow \mathbf{k}$ is called *symmetric* if

$$\varphi(v, w) = \varphi(w, v)$$

for all $v, w \in V$. A bilinear form is symmetric if and only if its matrix is symmetric. A *quadratic form* on V is a map $q : V \longrightarrow \mathbf{k}$ of the form $v \mapsto \varphi(v, v)$, where φ is a symmetric bilinear form. The form φ is uniquely determined by q as its *polarization*

$$(v, w)_q = (q(v + w) - q(v) - q(w))/2.$$

A quadratic form q is said to be *non-degenerate* if its polarization is non-degenerate.

The *Gram matrix* $G(q)$ of a quadratic form q is defined as the matrix of its polarization, and the determinant $\det q$ of q is defined as the determinant of its polarization, i.e.

$$\det q = \det G(q).$$

A quadratic form q is non-degenerate if and only if $\det q \neq 0$.

The determinant map. The set $Q(V)$ of quadratic forms on V is a vector space over \mathbf{k} . The map $q \mapsto G(q)$ is an isomorphism between $Q(V)$ and the space of symmetric $n \times n$ matrices with entries in \mathbf{k} . The determinants $\det q$ lead to the *determinant map*

$$\det : Q(V) \longrightarrow \mathbf{k}.$$

If \det' is the determinant map corresponding to some other basis of V , then

$$\det' q = \det q \cdot (\det A)^2,$$

where A is the matrix defined as above. It follows that up to a multiplication by a non-zero constant the determinant map is independent from the choice of basis of V .

Symmetric maps. A linear map $f: V \longrightarrow V$ is said to be *symmetric* with respect to a bilinear form φ if

$$\varphi(f(v), w) = \varphi(v, f(w))$$

for all $v, w \in V$. If φ is symmetric, then $\varphi(v, f(w)) = \varphi(f(w), v)$ for all $v, w \in V$. It follows that f is symmetric with respect to a symmetric bilinear form φ if and only if the left product $f \cdot \varphi$ is symmetric.

Quadratic forms and symmetric maps. A linear map $f: V \longrightarrow V$ is said to be *symmetric* with respect to a quadratic form q on V if f is symmetric with respect to the polarization of q , i.e. if the bilinear form

$$(v, w) \longmapsto (f(v), w)_q$$

is symmetric. If q is a quadratic form with the polarization φ and f is a linear map symmetric with respect to q , then the *product* $f \cdot q$ is defined as the quadratic form with the polarization $f \cdot \varphi$. Equivalently, the *product of f and q* is the map

$$f \cdot q: v \longmapsto (f(v), v)_q.$$

By applying (1.1) to f and the polarization φ of q , we see that

$$(1.2) \quad \det(f \cdot q) = \det f \cdot \det q.$$

Suppose that n is a non-degenerate quadratic form on V . Let p be an arbitrary quadratic form on V . Then there exist a unique linear map $f: V \longrightarrow V$ such that

$$(v, w)_p = (f(v), w)_n$$

for all $v, w \in V$. Since the polarization $(v, w)_p$ is symmetric, f is symmetric with respect to n . By taking $v = w$ in the last displayed formula, we see that $p = f \cdot n$. The map f is uniquely determined by the forms p, n . We will denote it by p/n .

The operations $p \longmapsto p/n$ and $f \longmapsto f \cdot n$ are mutually inverse in the sense that

$$p = (p/n) \cdot n \quad \text{and} \quad f = f \cdot n/n$$

for all $p \in Q(V)$ and all maps $f: V \longrightarrow V$ symmetric with respect to n .

Anisotropic forms. A quadratic form q is called *anisotropic* if $q(v) \neq 0$ for $v \neq 0$. Suppose that q is an anisotropic form with the polarization φ . If $v \neq 0$, then the map $w \longmapsto \varphi(v, w)$ is non-zero because $\varphi(v, v) = q(v) \neq 0$. It follows that $\hat{\varphi}$ is injective and hence is an isomorphism. Therefore φ and q are non-degenerate.

The case of vector spaces of dimension 2. Suppose now that the dimension of V is 2. Let v, w be a basis of V . If $q \in Q(V)$, then

$$G(q) = \begin{bmatrix} (v, v)_q & (v, w)_q \\ (w, v)_q & (w, w)_q \end{bmatrix}$$

is the Gram matrix of q with respect to the basis v, w . It follows that the determinant map $\det: Q(V) \rightarrow k$ with respect to the basis v, w has the form

$$\det q = (v, v)_q (w, w)_q - (v, w)_q^2 = q(v)q(w) - (v, w)_q^2.$$

Therefore the determinant map itself is a non-degenerate quadratic form.

Orthogonality. Two vectors $v, w \in V$ are said to be *orthogonal* with respect to a form $q \in Q(V)$ if $(v, w)_q = 0$, i.e. if

$$q(v + w) = q(v) + q(w).$$

Suppose now that the dimension of V is 2. Let p, q be two quadratic forms on V . They are said to be *orthogonal* if p, q are orthogonal with respect to a determinant map \det considered as a quadratic form on $Q(V)$, i.e. if

$$(1.3) \quad \det(p + q) = \det p + \det q.$$

Since different determinant maps differ by the multiplication by a non-zero element of k , the property (1.3) does not depend on the choice of the determinant map.

1.1. Theorem. Let V be a vector space over k of dimension 2 and let $p, q \in Q(V)$. Suppose that v, w is a basis of V orthogonal with respect to p . Then the forms p, q are orthogonal with respect to any determinant map if and only if

$$p(v)q(w) + q(v)p(w) = 0.$$

Proof. By the remark preceding the theorem, it is sufficient to consider the orthogonality with respect to determinant map defined by the basis v, w .

The forms p, q are orthogonal if and only if (1.3) holds, i.e. if and only if

$$\begin{aligned} (1.4) \quad (p + q)(v)(p + q)(w) - (v, w)_{p+q}^2 &= \\ &= (p(v)p(w) - (v, w)_p^2) + (q(v)q(w) - (v, w)_q^2). \end{aligned}$$

The basis v, w is orthogonal with respect to p and hence $(v, w)_p = 0$ and

$$(v, w)_{p+q} = (v, w)_p + (v, w)_q = (v, w)_q.$$

It follows that the (1.4) is equivalent to

$$(p+q)(v)(p+q)(w) - (v, w)_q^2 = p(v)p(w) + q(v)q(w) - (v, w)_q^2,$$

or, what is the same, to

$$(1.5) \quad (p+q)(v)(p+q)(w) = p(v)p(w) + q(v)q(w).$$

Since $(p+q)(u) = p(u) + q(u)$ for all $u \in V$, the left hand side of (1.5) is equal to

$$p(v)p(w) + q(v)q(w) + p(v)q(w) + q(v)p(w)$$

and hence (1.5) is equivalent to $p(v)q(w) + q(v)p(w) = 0$. The theorem follows. ■

2. The norm and the trace

The norm and the trace. Recall that $z \mapsto \bar{z}$ is the only non-trivial automorphism of the field K fixed on k . Let $N, \text{tr}: K \rightarrow k$ be the maps

$$N(z) = z\bar{z}, \quad \text{tr}(z) = (z + \bar{z})/2.$$

Then N is the norm of the extension K/k and tr is the half of the trace of K/k . The map tr has the advantage of being equal to the identity on k . If $K/k = \mathbb{C}/\mathbb{R}$, then \bar{z} is the complex conjugate of z , $\text{tr}(z) = \text{Re } z$ and $N(z) = |z|^2$.

The norm as a quadratic form on K . Let

$$\langle z, u \rangle = \text{tr}(z\bar{u}).$$

Then $\langle z, u \rangle$ is a bilinear form on K considered as a vector space over k . Since

$$\text{tr}(z\bar{u}) = \text{tr}(\overline{z\bar{u}}) = \text{tr}(u\bar{z})$$

for all $z, u \in K$, the bilinear form $\langle z, u \rangle$ is symmetric. Since $z\bar{z} = \text{tr}(z\bar{z})$ for all $z \in K$, it follows that N is a quadratic form on K with polarization $\langle z, u \rangle$. Obviously, the quadratic form N is anisotropic and hence is non-degenerate.

The orthogonal complement of \mathbf{k} . Let \mathbf{k}^\perp be the orthogonal complement of \mathbf{k} in \mathbf{K} with respect to N , i.e. the set of all $\rho \in \mathbf{K}$ such that $\langle \rho, \alpha \rangle = 0$ for all $\alpha \in \mathbf{k}$. Since N is anisotropic, $\mathbf{k} \cap \mathbf{k}^\perp = \emptyset$ and hence $\mathbf{K} = \mathbf{k} \oplus \mathbf{k}^\perp$. Since the dimension of \mathbf{K} over \mathbf{k} is equal to 2, the dimension of \mathbf{k}^\perp over \mathbf{k} is equal to 1.

The orthogonal complement \mathbf{k}^\perp is equal to the kernel of $\text{tr}: \mathbf{K} \rightarrow \mathbf{k}$, as it follows from the fact that $\langle \rho, \alpha \rangle = \alpha \text{tr}(\rho)$ for $\alpha \in \mathbf{k}$. Therefore $\rho \in \mathbf{k}^\perp$ implies that

$$\bar{\rho} = -\rho \quad \text{and} \quad \rho^2 = -\bar{\rho}\rho = -N(\rho) \in \mathbf{k}.$$

3. Quadratic forms in dimension 1 over \mathbf{K}

The framework. For the remaining part of the paper we will assume that V is a vector space over \mathbf{K} of dimension 1 over \mathbf{K} . By the restriction of scalars we may consider V also as a vector space over \mathbf{k} . By a quadratic form on V we will understand a quadratic form on V as a vector space over \mathbf{k} . As a vector space over \mathbf{k} the vector space $Q(V)$ of quadratic forms on V depends only on the structure of V as a vector space over \mathbf{k} .

The norm-like quadratic forms. A form $n \in Q(V)$ is said to be *norm-like* if

$$(3.1) \quad n(zv) = N(z)n(v)$$

for all $v \in V$ and $z \in \mathbf{K}$. Let $N(V)$ be the space of all norm-like forms on V .

The dimension of $N(V)$ over \mathbf{k} is equal to 1. Indeed, let $v \in V$ and $v \neq 0$. Then every element of V is equal to zv for some $z \in \mathbf{K}$ and hence every norm-like form is determined by its value at v . On the other hand, for every $\alpha \in \mathbf{k}$ the map $zv \mapsto N(z)\alpha$ is a norm-like form and takes the value α at v .

Every non-zero $n \in N(V)$ is anisotropic and hence is non-degenerate. Indeed, if $n(v) \neq 0$, then every non-zero element w of V is equal to zv for some non-zero $z \in \mathbf{K}$. Since N is anisotropic, together with (3.1) this implies that $n(w) \neq 0$.

The polarization of any norm-like form n is closely related to the polarization $\langle z, u \rangle$ of N . Let $v \in V$ and $z, u \in \mathbf{K}$. Then

$$(3.2) \quad (zv, uv)_n = \langle z, u \rangle n(v).$$

Indeed, the maps $z \mapsto n(zv)$ and $z \mapsto N(z)n(v)$ are quadratic forms on \mathbf{K} . By (3.1) they are equal. Hence their polarizations are equal also. But the left and the right hand sides of (3.2) are nothing else but these polarizations.

3.1. Lemma. *A quadratic form n on V is norm-like if and only if*

$$(3.3) \quad (zv, w)_n = (v, \bar{z}w)_n$$

for every $v, w \in V$ and every $z \in \mathbf{K}$.

Proof. Suppose that n is norm-like. We may assume that $v \neq 0$. Then v is a basis of V over \mathbf{K} and hence $w = uv$ for some $u \in \mathbf{K}$. The identity (3.2) implies that

$$(zv, w)_n = (zv, uv)_n = \langle z, u \rangle n(v) \quad \text{and}$$

$$(v, \bar{z}w)_n = (v, \bar{z}uv)_n = \langle 1, \bar{z}u \rangle n(v).$$

On the other hand, $\text{tr}(z\bar{u}) = \text{tr}(\bar{z}u) = \text{tr}(1 \cdot \bar{z}u)$ and hence

$$\langle z, u \rangle = \langle 1, \bar{z}u \rangle.$$

By combining these equalities we see that (3.3) holds. Conversely, if (3.3) holds, then

$$(zv, zv)_n = (v, \bar{z}zv)_n = z\bar{z}(v, v)_n$$

and hence $n(zv) = N(z)n(v)$ for all $z \in \mathbf{K}$, $v \in V$. ■

The anti-norm-like quadratic forms. A form $q \in Q(V)$ is said to be *anti-norm-like* if

$$(3.4) \quad q(\rho v) = -N(\rho)q(v) = \rho^2 q(v)$$

for all $v \in V$ and $\rho \in \mathbf{k}^\perp$. Let $\mathbf{A}(V)$ be the space of all anti-norm-like forms on V .

As in the case of norm-like forms, the identity (3.4) can be extended to the polarizations. Let q is an anti-norm-like form. Let $v, w \in V$ and let $\rho \in \mathbf{k}^\perp$. Then

$$(3.5) \quad (\rho v, \rho w)_q = -N(\rho)(v, w)_q = \rho^2(v, w)_q$$

Indeed, the maps $v \mapsto q(\rho v)$ and $v \mapsto -N(\rho)q(v)$ are quadratic forms on V . By (3.4) they are equal. Hence their polarizations are equal also. But the left and the right hand sides of (3.5) are nothing else but these polarizations.

3.2. Lemma. *A quadratic form q on V is anti-norm-like if and only if*

$$(3.6) \quad (zv, w)_q = (v, zw)_q$$

for every $v, w \in V$ and every $z \in \mathbf{K}$.

Proof. Suppose that q is anti-norm-like. Every $z \in \mathbf{K}$ has the form $z = a + \rho$, where $a \in \mathbf{k}$ and $\rho \in \mathbf{k}^\perp$. If $\rho = 0$, then $z = a \in \mathbf{k}$ and hence (3.6) holds. If $\rho \neq 0$, then ρ^{-1} is defined and (3.5) implies that

$$(\rho v, w)_q = \rho^2 (v, \rho^{-1} w)_q = (v, \rho^2 \rho^{-1} w)_q = (v, \rho w)_q.$$

On the other hand, $(av, w)_q = (v, aw)_q$ because $a \in \mathbf{k}$. Therefore

$$(av, w)_q + (\rho v, w)_q = (v, aw)_q + (v, \rho w)_q$$

It follows that $(zv, w)_q = (v, zw)_q$, i.e. the identity (3.6) holds.

Suppose now that (3.6) holds. If $v \in V$ and $\rho \in \mathbf{k}^\perp$, then

$$(\rho v, \rho v)_q = (\rho^2 v, v)_q = \rho^2 (v, v)_q$$

and hence $q(\rho v) = \rho^2 q(v)$. It follows that q is anti-norm-like. ■

3.3. Theorem. $\mathbf{A}(V)$ is equal to the orthogonal complement of $\mathbf{N}(V)$ in $\mathbf{Q}(V)$.

Proof. Suppose that $n \in \mathbf{N}(V)$, $n \neq 0$. Let $v \in V$, $v \neq 0$. Since n is non-zero, the property (3.1) implies that $n(v) \neq 0$. Let $\rho \in \mathbf{k}^\perp$. Then (3.2) implies that

$$(\rho v, v)_n = \langle \rho, 1 \rangle n(v) = \text{tr}(\rho) n(v) = 0$$

and hence ρv is orthogonal to v with respect to n .

By Theorem 1.1 the forms n, q are orthogonal if and only if

$$n(v)q(\rho v) + q(v)n(\rho v) = 0.$$

By the property (3.1) this equality is equivalent to

$$n(v)q(\rho v) + q(v)N(\rho)n(v) = 0.$$

Since $n(v) \neq 0$, the last equality is equivalent to

$$q(\rho v) + q(v)N(\rho) = 0$$

and hence to $q(\rho v) = -N(\rho)q(v)$.

It follows that n, q are orthogonal if and only if q is anti-norm-like. ■

3.4. Theorem. $Q(V) = N(V) \oplus A(V)$.

Proof. Suppose that $q \in N(V) \cap A(V)$. Let $\rho \in k^\perp$, $\rho \neq 0$. Then

$$q(\rho v) = N(\rho)q(v) \quad \text{and} \quad q(\rho v) = -N(\rho)q(v)$$

and hence $N(\rho)q(v) = 0$ for all $v \in V$. But if $\rho \neq 0$, then $N(\rho) \neq 0$. It follows that $q(v) = 0$ for all v , i.e. $q = 0$. Therefore

$$N(V) \cap A(V) = 0$$

In view of Theorem 3.3, this implies $Q(V) = N(V) \oplus A(V)$. ■

4. The structure of a K -vector space on $A(V)$

The multiplication maps. For $z \in K$ let $m_z: V \rightarrow V$ be the map

$$m_z: v \mapsto zv.$$

We consider m_z as a map linear over k (although it is linear over K also). Then

$$(4.1) \quad \det m_z = N(z).$$

If $V = K$, this is a well known result of the Galois theory. Since V is isomorphic to K as a vector space over K , this special case implies the general one.

The anti-norm-like quadratic forms and the multiplication maps. Suppose that q is an anti-norm-like quadratic form on V and let $z \in K$. Lemma 3.2 implies that m_z is symmetric with respect to q , and hence the product $m_z \cdot q$ is defined.

4.1. Lemma. *If q is an anti-norm-like quadratic form on V and $z \in K$, then the product $m_z \cdot q$ is also anti-norm-like.*

Proof. Let $\rho \in k^\perp$. By applying (3.5) to zv , v in the roles of v , w , we see that

$$\begin{aligned} (m_z \cdot q)(\rho v) &= (z(\rho v), \rho v)_q = (\rho(zv), \rho v)_q \\ &= -N(\rho)(zv, v)_q = -N(\rho)(m_z \cdot q)(v). \end{aligned}$$

It follows that $m_z \cdot q$ is anti-norm-like. ■

4.2. Lemma. If q is an anti-norm-like quadratic form on V and $z \in K$, then

$$\det(m_z \cdot q) = N(z) \det q$$

for every determinant map $\det: Q(V) \rightarrow k$.

Proof. It is sufficient to combine (1.2) with (4.1). ■

The structure of a vector space over K on $A(V)$. Let us define the multiplication $(z, q) \mapsto zq$ of forms $q \in A(V)$ by elements $z \in K$ by the formula

$$zq = m_{\bar{z}} \cdot q.$$

This turns $A(V)$ into a vector space over K , as a trivial verification shows. By Lemma 4.2 any determinant map defines a norm-like quadratic form on $A(V)$.

The multiplication $m_{\bar{z}}$ is used instead of the more natural m_z in order to avoid \bar{z} in Lemma 6.3. This makes the identification maps from Section 10 linear over K .

5. Conformal structures

Conformal structures. A *conformal structure* on a vector space U over k is defined as a non-zero quadratic form on U considered up to multiplication by a non-zero element of k . The conformal structure determined by $q \in Q(U)$ is called the *conformal class* of q and is denoted by $[q]$. It is called *non-degenerate* if the form q is non-degenerate. The set of all conformal structures on U is nothing else but the *projective space* $\mathbb{P}Q(U)$ associated with the vector space $Q(U)$ of quadratic forms on U .

Conformal structures in dimension 1 over K . The conformal class $c_V = [n]$ of a non-zero $n \in N(V)$ does not depend on the choice of n and is called the *canonical conformal structure* on V . A conformal structure on V is called *exceptional* if it is equal to the conformal class $[q]$ of a non-zero $q \in A(V)$, and *regular* otherwise.

Let $\mathbb{P}A(V) \subset \mathbb{P}Q(V)$ be the set of the exceptional conformal structures, i.e. be the set of conformal classes $[q]$ of quadratic forms $q \in A(V)$. Its complement

$$\mathbb{M}(V) = \mathbb{P}Q(V) \setminus \mathbb{P}A(V)$$

is the set of all regular conformal structures on V . The set $\mathbb{P}A(V)$ is a line in the projective plane $\mathbb{P}Q(V)$, and its complement $\mathbb{M}(V)$ is an affine plane.

The structure of a vector space over \mathbf{K} on $\mathbb{M}(V)$. Let $n \in \mathbf{N}(V)$, $n \neq o$. The map $q \mapsto [n + q]$ is a bijection $\mathbf{A}(V) \rightarrow \mathbb{M}(V)$. One can use it to transfer the structure of a vector space over \mathbf{K} from $\mathbf{A}(V)$ to $\mathbb{M}(V)$. The resulting multiplication by elements $z \in \mathbf{K}$ is given by the formula

$$z[n + q] = [n + zq].$$

While the bijection $\mathbf{A}(V) \rightarrow \mathbb{M}(V)$ depends on the choice of n , an immediate verification shows that the resulting multiplication does not. Therefore, the above construction defines a canonical structure of a vector space over \mathbf{K} on $\mathbb{M}(V)$. The canonical conformal structure c_V on V serves as the zero of this vector space.

A canonical quadratic form on $\mathbb{M}(V)$. Every point of $\mathbb{M}(V)$ has the form $[n + q]$, where $n \in \mathbf{N}(V)$, $n \neq o$, and $q \in \mathbf{A}(V)$. Let us define a map $\mathbb{D}: \mathbb{M}(V) \rightarrow \mathbf{k}$ by

$$(5.1) \quad \mathbb{D}: [n + q] \mapsto -\det q / \det n.$$

If $[n' + q'] = [n + q]$, then $n' = an$, $q' = aq$ for some $a \in \mathbf{k}$, and hence

$$\det q' / \det n' = \det q / \det n.$$

It follows that \mathbb{D} is correctly defined. If we temporarily fix n and identify $\mathbb{M}(V)$ with $\mathbf{A}(V)$ by the map $q \mapsto [n + q]$, then \mathbb{D} turns into the map $-\det/\det n$ restricted to $\mathbf{A}(V)$. Since \det is a quadratic form, this implies that \mathbb{D} is a quadratic form on $\mathbb{M}(V)$. Lemma 4.2 implies that \mathbb{D} is norm-like.

6. Anti-linear maps

Anti-linear maps. Let U, W be vector spaces over \mathbf{K} , and let f be a map $U \rightarrow W$ linear over \mathbf{k} . The map f is said to be *anti-linear* over \mathbf{K} , or simply *anti-linear* if

$$f(zu) = \bar{z} f(u)$$

for all $z \in \mathbf{K}$, $u \in U$. Let $\text{Hom}_a(U, W)$ be the space of all anti-linear maps $U \rightarrow W$.

6.1. Lemma. *If $f \in \text{Hom}_a(V, V)$ and $n \in \mathbf{N}(V)$, then f is symmetric with respect to n .*

Proof. If $v, w \in V$ and $v \neq o$, then $w = zv$ for some $z \in \mathbf{K}$, and

$$(f(v), zv)_n = (\bar{z} f(v), v)_n = (f(zv), v)_n$$

by Lemma 3.1 and the anti-linearity of f . It follows that

$$(f(v), w)_n = (f(w), v)_n = (v, f(w))_n$$

and hence f is symmetric with respect to n . ■

6.2. Lemma. *Let $n \in \mathbf{N}(V)$, $n \neq 0$. If $f: V \rightarrow V$ is anti-linear, then $f \cdot n$ is defined and $f \cdot n \in \mathbf{A}(V)$. Conversely, if $q \in \mathbf{A}(V)$, then $f = q/n$ is anti-linear.*

Proof. If $f: V \rightarrow V$ is anti-linear, then by Lemma 6.1 f is symmetric with respect to n and hence $f \cdot n$ is defined. Let $q = f \cdot n$. If $z \in \mathbf{K}$ and $v, w \in V$, then

$$(\bar{z} f(v), w)_n = (f(zv), w)_n = (zv, w)_q$$

by anti-linearity of f and the definition of q , and

$$(\bar{z} f(v), w)_n = (f(v), zw)_n = (v, zw)_q$$

by Lemma 3.1 and the definition of q . It follows that $(zv, w)_q = (v, zw)_q$ for all $z \in \mathbf{K}$ and all $v, w \in V$. By Lemma 3.2 this implies that $f \cdot n = q \in \mathbf{A}(V)$.

Suppose now that $q \in \mathbf{A}(V)$. Since n is non-degenerate, the map $f = q/n: V \rightarrow V$ is defined and is symmetric with respect to n . If $z \in \mathbf{K}$ and $v, w \in V$, then

$$(zv, w)_q = (v, zw)_q$$

by Lemma 3.2 and hence $(f(zv), w)_n = (f(v), zw)_n$ by the definition of f . Since

$$(f(v), zw)_n = (\bar{z} f(v), w)_n$$

by Lemma 3.1, it follows that

$$(f(zv), w)_n = (\bar{z} f(v), w)_n$$

for all $z \in \mathbf{K}$ and all $v, w \in V$. Since n is non-degenerate and w is arbitrary, it follows that $f(zv) = \bar{z} f(v)$ for all $z \in \mathbf{K}$, $v \in V$, i.e. that f is anti-linear. ■

6.3. Lemma. *Let $n \in \mathbf{N}(V)$, $n \neq 0$. If $f \in \text{Hom}_a(V, V)$ and $z \in \mathbf{K}$, then*

$$(zf) \cdot n = z(f \cdot n).$$

Proof. Let $q = f \cdot n$ and $p = (zf) \cdot n$. Let $v, w \in V$. Then

$$(v, w)_{zq} = (m_{\bar{z}}(v), w)_q = (\bar{z}v, w)_q$$

by the definition of the product zq . On the other hand, $(zf)(v) = zf(v) = f(\bar{z}v)$ by the definition of the product zf and the anti-linearity of f and hence

$$(v, w)_p = (f(\bar{z}v), w)_n = (\bar{z}v, w)_q$$

by the definition of p and q . It follows that $p = zq$, i.e. $(zf) \cdot n = z(f \cdot n)$. ■

7. The structure of $\text{Hom}_a(V, V)$

The structure of a vector space over \mathbf{K} on $\text{Hom}_a(V, V)$. The product zf of an element $z \in \mathbf{K}$ with $f \in \text{Hom}_a(V, V)$ is defined by the formula

$$zf = m_z \circ f.$$

Obviously, this operation turns $\text{Hom}_a(V, V)$ into a vector space over \mathbf{K} . Since the dimension of V over \mathbf{K} is n , an anti-linear map $V \rightarrow V$ is determined by its value on any given non-zero $v \in V$. It follows that the dimension of $\text{Hom}_a(V, V)$ over \mathbf{K} is also n , and hence any non-zero $g \in \text{Hom}_a(V, V)$ is a basis of $\text{Hom}_a(V, V)$.

The composition $(f, g) \mapsto f \circ g$ is a *hermitian map*

$$\text{Hom}_a(V, V) \times \text{Hom}_a(V, V) \rightarrow \text{Hom}_a(V, V)$$

in the sense that it is bilinear and $(zf) \circ (ug) = z\bar{u}(f \circ g)$ for all $z, u \in \mathbf{K}$.

A canonical bilinear form on $\text{Hom}_a(V, V)$. Given $f, g \in \text{Hom}_a(V, V)$, let

$$\langle f, g \rangle = \text{Tr}(f \circ g)/2,$$

where the trace is taken over \mathbf{k} . Clearly, $\langle f, g \rangle$ is a bilinear form and since

$$\text{Tr}(f \circ g) = \text{Tr}(g \circ f)$$

for all maps $f, g: V \rightarrow V$ linear over \mathbf{k} , this bilinear form is symmetric. Therefore

$$\mathcal{D}: f \mapsto \langle f, f \rangle/2$$

is a quadratic form on $\text{Hom}_a(V, V)$ having the form $\langle f, g \rangle$ as its polarization.

Reflections. Let σ be the map $z \mapsto \bar{z}$. Then σ is an anti-linear map $\mathbf{K} \rightarrow \mathbf{K}$ and $\sigma \circ \sigma = \text{id}_{\mathbf{K}}$. Since $\sigma(a) = a$ for $a \in \mathbf{k}$ and $\sigma(\rho) = -\rho$ for $\rho \in \mathbf{k}^\perp$, the determinant $\det \sigma$ over \mathbf{k} is equal to -1 .

Since V is isomorphic to \mathbf{K} as a vector space over \mathbf{K} , it follows that there exist anti-linear maps $g: V \rightarrow V$ such that $g \circ g = \text{id}_V$ and $\det g = -1$. We will call any such anti-linear map $V \rightarrow V$ a *reflection*.

7.1. Lemma. *If $f \in \text{Hom}_a(V, V)$, then $f \circ f = \mathcal{D}(f) \text{id}_V$ and $\det f = -\mathcal{D}(f)$, where the determinant is taken over \mathbf{k} .*

Proof. Let $g \in \text{Hom}_a(V, V)$ be a reflection. Then $g \neq 0$ and hence g is a basis of $\text{Hom}_a(V, V)$. It follows that $f = zg$ for some $z \in \mathbf{K}$ and hence

$$f \circ f = (zg) \circ (zg) = z\bar{z} (g \circ g) = N(z) \text{id}_V.$$

Therefore $\mathcal{D}(f) = \text{Tr}(f \circ f)/2 = N(z)$. It follows that $f \circ f = \mathcal{D}(f) \text{id}_V$.

On the other hand, $f = m_z \circ g$ and hence Lemma 4.2 implies that

$$\det f = (\det m_z) \cdot (\det g) = N(z) \cdot (-1) = -N(z).$$

Since we already proved that $\mathcal{D}(f) = N(z)$, it follows that $\det f = -\mathcal{D}(f)$. ■

8. The conformal dilatation and Beltrami forms

Pull-backs of quadratic forms. Let U, U' be vector spaces over \mathbf{k} and $f: U \rightarrow U'$ be a linear map. Let $q \in Q(U')$. The *pull-back* $f^*q \in Q(U)$ is the quadratic form

$$f^*q: u \mapsto q(f(u)).$$

The pull-backs f^*q lead to a linear map $f^*: Q(U') \rightarrow Q(U)$. If U'' is one more vector space over \mathbf{k} and $g: U' \rightarrow U''$ is a linear map, then $(g \circ f)^* = f^* \circ g^*$. If f is an isomorphism, then f^* is an isomorphism also.

If $p \in Q(U')$ and $f^*p \neq 0$, then the conformal structure $[f^*p]$ depends only on $[p]$. It is called the *pull-back* of p by f and is denoted by $f^*[p]$.

The conformal dilatation. Let W be another vector space of dimension 1 over \mathbf{K} and let $f: V \rightarrow W$ be a non-zero map linear over \mathbf{k} . The pull-back

$$c(f) = f^*c_W \in \mathbb{P}Q(V)$$

of the canonical conformal structure c_W on W is called the *conformal dilatation* of f .

The map f is called *exceptional* if the conformal structure $c(f)$ is exceptional, and *regular* otherwise. If f is regular, then we consider $c(f)$ as an element of $\mathbb{M}(V)$.

If the field extension \mathbf{K}/\mathbf{k} resembles enough the classical case \mathbf{C}/\mathbf{R} , then all non-zero f are regular. For example, by using Theorem 10.2 it is not hard to prove that this is the case if \mathbf{k} is an ordered field and $\mathbf{K} = \mathbf{k}(\rho)$ with $\rho^2 < 0$.

The map f is called *conformal* if the pull-back by f of the canonical conformal structure on W is equal to the canonical conformal structure on V , i.e. if

$$f^*c_W = c_V.$$

Since c_V serves as the zero of $\mathbb{M}(V)$, the map f is conformal if and only if f is regular and $c(f) = 0 \in \mathbb{M}(V)$. In general, the conformal dilatation of f is a natural measure of the distortion of the canonical conformal structure by f .

Linear maps over \mathbf{k} as sums of linear and anti-linear over \mathbf{K} maps. Every map $f: V \rightarrow W$ linear over \mathbf{k} admits a unique presentation as a sum

$$f = Lf + Af$$

of a map Lf linear over \mathbf{K} and a map Af anti-linear over \mathbf{K} . Indeed, let $\rho \in \mathbf{k}^\perp$, $\rho \neq 0$. If $f = Lf + Af$ is such a presentation of f , then

$$f(v) = Lf(v) + Af(v), \quad f(\rho v) = \rho Lf(v) - \rho Af(v),$$

and hence

$$Lf(v) = (f(v) - \rho^{-1}f(\rho v))/2, \quad Af(v) = (f(v) + \rho^{-1}f(\rho v))/2$$

for all $v \in V$. This proves uniqueness. Conversely, one can define Lf and Af by the last displayed formulas. Since \mathbf{k}^\perp has dimension 1 over \mathbf{k} , this definition does not depend on the choice of ρ . We leave to the reader the verification that so defined maps Lf and Af are linear and anti-linear over \mathbf{K} respectively.

Beltrami forms. Let $f: V \rightarrow W$ is a map linear over \mathbf{k} such that $Lf \neq 0$. Then

$$\mu_f = (Lf)^{-1}(Af) \in \text{Hom}_a(V, V)$$

is called the *Beltrami form* of f . The map f is linear over \mathbf{K} if and only if $\mu_f = 0$. In general, the Beltrami form of f is a natural measure of deviation of f from being linear over \mathbf{K} . It is a direct generalization of the classical Beltrami forms from the theory of quasi-conformal mappings. See, for example, [H], Section 4.8.

9. An identification of $\mathbb{M}(V)$ with $\text{Hom}_a(V, V)$

An identification of $\mathbb{M}(V)$ with $\text{Hom}_a(V, V)$. Every conformal structure in $\mathbb{M}(V)$ has the form $[n + q]$ for some non-zero $n \in \mathbf{N}(V)$ and some $q \in \mathbf{A}(V)$. Let

$$(9.1) \quad \mathbb{M}(V) \longrightarrow \text{Hom}_a(V, V)$$

be the map defined by the rule $[n + q] \mapsto q/n$. Since, obviously, $aq/an = q/n$ for all non-zero $a \in \mathbf{k}$, this map is well defined.

Let us choose some $n \in \mathbf{N}(V)$, $n \neq 0$. Let

$$(9.2) \quad \text{Hom}_a(V, V) \longrightarrow \mathbb{M}(V)$$

be the map defined by the rule $f \mapsto [n + f \cdot n]$. Since, obviously, $f \cdot (an) = a(f \cdot n)$ and hence $[an + f \cdot (an)] = [a(n + f \cdot n)] = [n + f \cdot n]$ for all non-zero $a \in \mathbf{k}$, this map in fact does not depend on the choice of n .

Since the operations $q \mapsto q/n$ and $f \mapsto f \cdot n$ are mutually inverse, these two maps between $\mathbb{M}(V)$ and $\text{Hom}_a(V, V)$ are mutually inverse bijections. Lemma 6.3 implies that the second map is linear over \mathbf{K} . It follows that the first map is linear over \mathbf{K} also, and hence both of these maps are isomorphisms of vector spaces over \mathbf{K} . We will identify $\mathbb{M}(V)$ and $\text{Hom}_a(V, V)$ by these isomorphisms.

The identification of the quadratic forms \mathbb{D} and \mathcal{D} . Let $\det: Q(V) \rightarrow \mathbf{k}$ be a determinant map. Let $n \in \mathbf{N}(V)$, $n \neq 0$. Suppose that $f \in \text{Hom}_a(V, V)$ and let $q = f \cdot n$. Then f is identified with the conformal class $[n + f \cdot n] = [n + q]$.

By applying (1.2) to f and to n in the role of q , we see that $\det q = \det f \cdot \det n$. Since $\det f = -\mathcal{D}(f)$ by Lemma 7.1, it follows that

$$\mathcal{D}(f) = -\det q / \det n.$$

On the other hand, by the definition (5.1) the quotient $-\det q / \det n$ is nothing else but the value of \mathbb{D} on the conformal class $[n + q]$ identified with f .

It follows that the pull-back of the quadratic form \mathcal{D} by the map (9.2) is equal to \mathbb{D} . Since the map (9.1) is equal to the inverse of (9.2), this implies that also the pull-back of the quadratic form \mathbb{D} by the map (9.1) is equal to \mathcal{D} .

In other words, the identification of $\mathbb{M}(V)$ and $\text{Hom}_a(V, V)$ identifies the canonical quadratic forms \mathbb{D} and \mathcal{D} on these spaces.

10. Comparing $\mathbf{c}(f)$ and μ_f

The framework. Let V, W be two vector spaces over \mathbf{K} , and let $f: V \rightarrow W$ be a map linear over \mathbf{k} . Suppose that $Lf \neq 0$ and let $\mu = \mu_f$ be the Beltrami form of f .

10.1. Lemma. *The conformal dilatation $\mathbf{c}(f)$ of f is equal to the conformal class of*

$$(10.1) \quad (1 + \mathcal{D}(\mu))n + 2\mu \cdot n$$

for any non-zero $n \in \mathbf{N}(V)$.

Proof. Since the conformal class of $(1 + \mathcal{D}(\mu))n + 2\mu \cdot n$ does not depend on the choice of $n \neq 0$, it is sufficient to prove the claim only for one particular $n \neq 0$.

Let $m \in \mathbf{N}(W)$, $m \neq 0$. By the definition, $\mathbf{c}(f)$ is equal to the conformal class of f^*m . Let $n = (Lf)^*(m)$. Then $n \in \mathbf{N}(V)$ because Lf is linear over \mathbf{K} and $n \neq 0$ because $Lf \neq 0$ and m is anisotropic. It follows that $(Lf^{-1})^*n = m$ and hence

$$\begin{aligned} f^*m &= (Lf)^*m + (Af)^*m \\ &= (Lf)^*(Lf^{-1})^*n + (Af)^*(Lf^{-1})^*n \\ &= n + (Lf^{-1} \circ Af)^*n = n + \mu^*n = (\text{id} + \mu)^*n. \end{aligned}$$

Let $p = f^*m$. By the above calculation $p = (\text{id} + \mu)^*n$ and hence

$$\begin{aligned} (v, w)_p &= ((\text{id} + \mu)(v), (\text{id} + \mu)(w))_n \\ &= (v + \mu(v), v + \mu(w))_n \\ &= (v, w)_n + (\mu(v), \mu(v))_n + (v, \mu(w))_n + (\mu(v), w). \end{aligned}$$

By Lemma 6.1 the map μ is symmetric with respect to n . It follows that

$$(v, w)_p = (v, w)_n + (\mu \circ \mu(v), w)_n + 2(\mu(v), w)_n.$$

Lemma 7.1 implies that $\mu \circ \mu = \mathcal{D}(\mu) \text{id}_V$ and hence

$$(v, w)_p = (v, w)_n + \mathcal{D}(\mu)(v, w)_n + 2(\mu(v), w)_n.$$

It follows that $f^*m = p = (1 + \mathcal{D}(\mu))n + 2\mu \cdot n$. Since $\mathbf{c}(f)$ is equal to the conformal class of f^*m , this proves the lemma. ■

10.2. Theorem. *If $1 + \mathcal{D}(\mu) \neq 0$, then f is regular, $c(f) \in \mathbb{M}(V)$, and*

$$c(f) = \frac{2\mu}{1 + \mathcal{D}(\mu)}$$

after the identification of $\mathbb{M}(V)$ with $\text{Hom}_a(V, V)$.

Proof. If $1 + \mathcal{D}(\mu) \neq 0$, then the quadratic form (10.1) does not belong to $\mathbf{A}(V)$ and the conformal class of the form (10.1) is equal to the conformal class of

$$(10.2) \quad n + \frac{2\mu}{1 + \mathcal{D}(\mu)} \cdot n.$$

In view of Lemma 10.1, this implies that $c(f) \in \mathbb{M}(V)$ and $c(f)$ is equal to the conformal class of (10.2). It remains to notice that conformal class is identified with

$$\frac{2\mu}{1 + \mathcal{D}(\mu)} \in \text{Hom}_a(V, V). \quad \blacksquare$$

10.3. Theorem. *If $1 + \mathcal{D}(\mu) = 0$, then f is exceptional, $c(f) \in \mathbb{PA}(V)$, and*

$$c(f) = [2\mu \cdot n]$$

for any non-zero $n \in \mathbf{N}(V)$.

Proof. Since $2\mu \cdot n \in \mathbf{A}(V)$ by Lemma 6.2, this follows from Lemma 10.1. \blacksquare

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